

Non-Relativistic Spacetimes with Cosmological Constant

R. Aldrovandi, A. L. Barbosa, L. C. B. Crispino* and J. G. Pereira

*Instituto de Física Teórica
Universidade Estadual Paulista
Rua Pamplona 145
01405-900 São Paulo
Brazil*

Abstract

Recent data on supernovae favor high values of the cosmological constant. Spacetimes with a cosmological constant have non-relativistic kinematics quite different from Galilean kinematics. De Sitter spacetimes, vacuum solutions of Einstein's equations with a cosmological constant, reduce in the non-relativistic limit to Newton-Hooke spacetimes, which are non-metric homogeneous spacetimes with non-vanishing curvature. The whole non-relativistic kinematics would then be modified, with possible consequences to cosmology, and in particular to the missing-mass problem.

I. INTRODUCTION

Recent data on supernovae, coming from two independent programs¹ and favoring high values of the cosmological constant, have renewed the interest on the non-relativistic limits of the corresponding spacetimes. Spacetimes with a cosmological constant have non-relativistic kinematics quite different from Galilean kinematics. They can, therefore, modify the non-relativistic physics with further implications to cosmology, as for example in the use of the virial theorem in connection with the missing-mass problem in galaxy clusters. Non-relativistic spacetimes have been studied in detail years ago.² We present here another approach to the problem, using the technique of group contraction to obtain non-relativistic kinematics from relativistic kinematics. The coordinate system required by the procedure, which reduces to Galilean coordinates in the appropriate limit, makes this approach nearer to observational practice.

As solutions of the sourceless Einstein's equations with a cosmological term, de Sitter spacetimes will play a fundamental role. However, as we shall be making good use of homogeneous spaces in our approach, it will be convenient to start our discussion with Lorentzian kinematics. Despite its trivial (that is, flat) connection, and as the simplest

*Permanent address: Departamento de Física, Universidade Federal do Pará, Campus Universitário do Guamá, 66075-900 Belém PA, Brazil.

(flat, vacuum) solution of Einstein's equations, Minkowski spacetime M is a true, even paradigmatic spacetime. It is taken as the local, "kinematical" spacetime and can also be identified with the space tangent to (real, curved) spacetime at each point. Above all, it has a consistent kinematics, in the sense that its metric is invariant under the appropriate kinematical (Poincaré) group P . This group contains the Lorentz group $L = SO(3, 1)$ and includes the translation subgroup T , which acts transitively on M and is, in a sense, its "double". Indeed, Minkowski spacetime appears as a homogeneous space under P , actually as the quotient $M = T = P/L$. If we prefer, the manifold of P is a principal bundle $(P/L, L)$ with $T = M$ as base space and L as the typical fiber.

The invariance of M under the transformations of P reflects its uniformity. Also in this "Copernican" aspect, Minkowski spacetime establishes a paradigm. P has the maximum possible number of Killing vectors, which is ten for a 4-dimensional flat spacetime. The Lorentz subgroup provides an isotropy around a given point of M , and the translation invariance enforces this isotropy around any other point. This is the meaning of "uniformity": all the points of spacetime are ultimately equivalent.

The reduction of relativistic to Galilean kinematics in the non-relativistic limit is the standard example of Inönü–Wigner contraction,³ by which the Poincaré group is contracted to the Galilei group. However, if we insist on the central role of the metric, there is no such a thing as a real "Galilean spacetime". The original metric is somehow "lost" in the process of contraction, and no metric exists which is invariant under the Galilei group. Minkowski spacetime tends, in the non-relativistic limit, to something that is not a spacetime. Nevertheless, there exists a meaningful connection which survives, even though the metric becomes undefined. This is not easily visible in the Minkowski–Galilei case, because both the initial and the final connections are flat.

Actually, in all local, or tangential physics, what happens is that the laws of Physics are invariant under transformations related to an uniformity as that described above. It includes homogeneity of space and of time, isotropy of space and the equivalence of inertial frames. This holds for Galilean and for special-relativistic physics, their difference being grounded only in their different "kinematical groups". However, as was clearly shown by Bacry and Lévy–Leblond,⁴ the corresponding Galilei and Poincaré groups are not the only ones to answer these uniformity requirements. Other groups, like the de Sitter groups⁵ and their non-relativistic Inönü–Wigner contractions, the so called Newton–Hooke groups,⁶ are in principle acceptable candidates.

The complete kinematical group, whatever it may be, will always have a subgroup accounting for the isotropy of space (rotation group) and the equivalence of inertial frames (boosts of different kinds for each case). The remaining transformations are "translations", which may be either commutative or not. Roughly speaking, the point-set of the corresponding spacetime is, in each case, the point-set of these translations. More precisely, kinematical spacetime is defined as the quotient space of the whole kinematical group by the subgroup including rotations and boosts. This means that local spacetime is always a homogeneous manifold.

Amongst curved spacetimes, only those of constant curvature share with Minkowski spacetime the property of lodging the highest number of Killing vectors. Given the metric signature and the value of the scalar curvature R , these maximally-symmetric spaces are unique.⁷ In consequence, the de Sitter spacetimes are the only uniformly curved 4-

dimensional metric spacetimes. There are two kinds of them,⁸ both conformally flat. One of them, the de Sitter spacetime proper, has the pseudo-orthogonal group $SO(4, 1)$ as group of motions and will be denoted $DS(4, 1)$. The other is the anti-de Sitter spacetime. It will be denoted $DS(3, 2)$ because its group of motions is $SO(3, 2)$. They are both homogeneous spaces: $DS(4, 1) = SO(4, 1)/SO(3, 1)$ and $DS(3, 2) = SO(3, 2)/SO(3, 1)$. The manifold of each de Sitter group is a bundle with the corresponding de Sitter spacetime as base space and $SO(3, 1)$ as fiber.

The de Sitter spacetimes are vacuum solutions of Einstein's equations with a cosmological constant and, as such, valid alternatives to Minkowski spacetime as local kinematical spacetimes. Provided the de Sitter pseudo-radius parameter L (or inverse cosmological constant) be large enough, it becomes impossible to know whether the true local relativistic group is the Poincaré group or one of the de Sitter groups, as no experiment could distinguish between their consequences. However, if the recent cosmological data favoring low values of L comes to be confirmed, non-relativistic kinematics would be governed by a Newton-Hooke group, not by the Galilei group.

We shall thus be concerned with such very special kinds of spacetime, the homogeneous spacetimes of groups which can be called kinematical. Actually, we shall be mainly interested in the Newton-Hooke cases, but its study will require a previous treatment of the Poincaré and de Sitter cases, the Galilei case coming as a corollary. A point we wish to emphasize is that the general theory of homogeneous spaces warrants the presence of a connection on such spaces, even when a metric is absent. Applied to our case, we shall see that the theory endorses the common knowledge on the Minkowski and de Sitter spacetimes, with a metric and corresponding connection (with vanishing curvature for Minkowski, but not for de Sitter spacetimes). In the Galilei case, it yields no metric and a flat connection. In the Newton-Hooke cases it also gives no metric, but gives a connection with non-vanishing curvature.

As all cases of our concern will be obtained from the de Sitter cases by convenient Inönü-Wigner contractions, our main tools will be the de Sitter groups and spacetimes. For example, the Galilei group is obtained from the de Sitter groups by two contractions: one which is a non-cosmological ($L \rightarrow \infty$) limit, and another one which is a non-relativistic ($c \rightarrow \infty$) limit. The Newton-Hooke group, on the other hand, is obtained from the de Sitter groups by taking only the non-relativistic limit.

Newton-Hooke spacetimes yield the non-relativistic kinematics in the case of an eventual non-vanishing cosmological constant, and may open the way to some new experimental test allowing to figure out the value of L . From a more theoretical point of view, the de Sitter spacetimes leading to the Newton-Hooke cases can appear in two different situations: (i) as spacetime itself, and (ii) as the tangent space to spacetime. The latter comes up in gauge theories for the de Sitter group, in which the fiber is a de Sitter spacetime tending to the tangent space as $L \rightarrow \infty$. Gauge theories for the Poincaré group⁹ are, basically, not quantizable, but this difficulty is solved if the Poincaré group is replaced by a de Sitter group.¹⁰ From this (gauge, quantum) point of view, the latter are preferable.

The theory of homogeneous spaces uses mainly the group Lie algebra, that is, the algebra of left-invariant fields on the group manifold. The general theory¹¹ says much more than what is necessary for our purposes. It contains a whole treatment of invariant connections, and we only shall need the so called "canonical" connections. We begin in section 2 with

a *resumé* on what the theory of homogeneous spaces says about those particular connections. To make the formalism easier to follow, we keep the symbols “ P ” for the general kinematical group and “ T ” for the translational sector, whatever it may be. Section 3 is a quick presentation of the conditions for an invariant metric to exist. We introduce de Sitter spacetimes, and apply the algebraic formalism to them, in section 4. We find that, for de Sitter spacetimes, the canonical connection is just the usual, metric connection. We find the curvature, but in the Maurer–Cartan basis. The tetrad field is then used to obtain the spacetime curvature from its corresponding “algebraic” expression. The Newton–Hooke cases are then presented in section 5. It is shown that, though no invariant metric exists, a non-trivial invariant connection is well-defined, and the non-vanishing components of the Riemann tensor are computed. Some aspects of Newton–Hooke physics are presented in the end of that section. Section 6 is dedicated to our final remarks.

II. THE GENERAL ALGEBRAIC SCHEME

Given a connected Lie group P and a closed subgroup L of P , a homogeneous space is defined by the quotient P/L . Consider the Lie algebras \mathcal{P} of P and \mathcal{L} of L . It will be supposed that a subspace \mathcal{T} of \mathcal{P} exists such that the underlying vector space of \mathcal{P} is the direct sum $\mathcal{P} = \mathcal{L} + \mathcal{T}$, and \mathcal{T} is invariant under the adjoint action of \mathcal{L} . All this means that \mathcal{P} has a multiplication table of the form

$$[\mathcal{L}, \mathcal{L}] \subset \mathcal{L} ; \quad (1)$$

$$[\mathcal{L}, \mathcal{T}] \subset \mathcal{T} ; \quad (2)$$

$$[\mathcal{T}, \mathcal{T}] \subset \mathcal{P} = \mathcal{L} + \mathcal{T} . \quad (3)$$

This type of Lie algebra, and also the resulting homogeneous space, will be called “reductive”.¹² This particular case of homogeneous space includes all spacetimes usually defined from kinematical groups. Properties (1) and (3) are essential to an algebraic discussion of spacetime, as translations do not constitute a subgroup in the general case and L can be seen as a typical fiber.

With our interest in spacetimes in mind, we shall take double-indexed operators $\{L_{\alpha\beta}\}$ for the generators of L and simply-indexed $\{T_\gamma\}$ for those of T . We shall be using the first half of the Greek alphabet ($\alpha, \beta, \gamma, \dots = 0, 1, 2, 3$) to denote algebraic indices, and the second half ($\lambda, \mu, \nu, \dots = 0, 1, 2, 3$) to denote spacetime indices. Thus, the above commutation rules will be

$$[L_{\gamma\delta}, L_{\epsilon\phi}] = \frac{1}{2} f^{(\alpha\beta)}_{(\gamma\delta)(\epsilon\phi)} L_{\alpha\beta} ; \quad (4)$$

$$[L_{\gamma\delta}, T_\epsilon] = f^{(\alpha)}_{(\gamma\delta)(\epsilon)} T_\alpha ; \quad (5)$$

$$[T_\gamma, T_\epsilon] = \frac{1}{2} f^{(\alpha\beta)}_{(\gamma)(\epsilon)} L_{\alpha\beta} + f^{(\alpha)}_{(\gamma)(\epsilon)} T_\alpha \quad (6)$$

(the factors $1/2$ account for repeated double-indices). Expression (2) or (5) says that the T ’s belong to a vector representation of L . They are really vectors if they belong to a commutative algebra. If, as allowed above, they do not, they will be “extended”, that is, they will include a “connection” with a curvature along L , and torsion along T .

The canonical form of P will be $\omega = (1/2)L_{\alpha\beta}\omega^{\alpha\beta} + T_\gamma\omega^\gamma$, where the $\omega^{\alpha\beta}$'s and ω^γ 's are the Maurer–Cartan forms, dual to the generators. The above commutation relations are equivalent to their dual versions, the Maurer–Cartan equations, which can be stated as:

$$d\omega + \omega \wedge \omega = 0 . \quad (7)$$

For us, the most important result of the theory is the following: the \mathcal{L} -component of the canonical form of P in that decomposition defines a connection Γ in the bundle $(P/L, L)$ which is invariant under the left-action of P :

$$\Gamma = \frac{1}{2}(L_{\gamma\delta})\omega^{\gamma\delta} . \quad (8)$$

The general theory allows many other connections, but this one (called “canonical”) is the most interesting, because of its many fair properties:

- (i) it always exists in the reductive case;
- (ii) its geodesics are the exponentials of straight lines on the tangent space;
- (iii) it is a complete connection;
- (iv) its curvature and torsion are parallel-transported;
- (v) it transports parallelly any P -invariant tensor.

Property (i) is the most important for our considerations. The matrix elements of Γ will be

$$\Gamma^\alpha{}_\beta = \frac{1}{2}f^{(\alpha)}{}_{(\gamma\delta)(\beta)}\omega^{\gamma\delta} . \quad (9)$$

The underlying vector space of \mathcal{T} will be identified with the horizontal space at the identity, as it is the set of vector fields X such that $\Gamma(X) = 0$. We shall identify the homogeneous space to T . It should be stressed that (8) is a Lie-algebra-valued form. The generators $L_{\gamma\delta}$ must be taken in the representation of interest. On the vector space of the T_α 's, they act as matrices, whose matrix elements are shown in (9). With this identification, a basis for the forms on T will be given by the ω^α 's, and we shall use the notation $h = T_\alpha\omega^\alpha$ for the “horizontal” part of the canonical form.

The curvature form R of a connection Γ is given by

$$R = D_\Gamma\Gamma = d\Gamma + \Gamma \wedge \Gamma .$$

For the case of reductive homogeneous spacetimes, it is fixed by $R(X, Y)Z = -[[X, Y]_{\mathcal{L}}, Z]$ (where $[,]_{\mathcal{L}}$ denotes the \mathcal{L} -component of the commutator) for any left-invariant fields $X, Y, Z \in \mathcal{T}$. Writing the curvature form R as

$$R = \frac{1}{4}L_{\alpha\beta}R^{\alpha\beta}{}_{\gamma\delta}\omega^\gamma \wedge \omega^\delta ,$$

we obtain for its components

$$R^{\alpha\beta}{}_{\gamma\delta} = -f^{(\alpha\beta)}{}_{(\gamma)(\delta)} . \quad (10)$$

The torsion of a connection Γ is given by the covariant derivative of the basis h according to Γ :

$$\Theta = D_\Gamma h = dh + \Gamma \wedge h + h \wedge \Gamma .$$

For reductive homogeneous spacetimes, the torsion is fixed by $\Theta(X, Y) = -[X, Y]_\mathcal{T}$, so that a non-vanishing torsion requires that some $f^{(\alpha)}_{(\gamma)(\delta)} \neq 0$. Writing the torsion form Θ as

$$\Theta = \frac{1}{2} T_\alpha \Theta^\alpha_{\gamma\delta} \omega^\gamma \wedge \omega^\delta ,$$

we obtain for its components

$$\Theta^\alpha_{\gamma\delta} = -f^{(\alpha)}_{(\gamma)(\delta)} . \quad (11)$$

Homogeneous spacetimes with torsion can appear when invariance under parity and time reversal transformations are ignored.¹³

It should be said that, conversely, any connection on the bundle $(P/L, L)$, which is invariant under the left-action of P , determines a decomposition as above.

The general properties can therefore be obtained by inspecting the multiplication table. One drawback of the theory is that the connection and its curvature and torsion come out naturally in a particular basis, that constituted by the Maurer–Cartan forms of the whole group.

III. INVARIANT METRICS

The above connection is invariant under the action of P . However, in the general case, it has nothing to do with a metric. The scheme does provide for metrics, and establishes conditions under which the connection is metric. Actually there will be a (P -invariant) metric on T for each ad_L -invariant non-degenerate symmetric bilinear form B on P/L : $g(X, Y) = B(X, Y)$ for all $X, Y \in \mathcal{T}$. The required invariance under ad_L is expressed by

$$B([Z, X], Y) + B(X, [Z, Y]) = 0 ,$$

for all $X, Y \in \mathcal{T}$ and $Z \in \mathcal{L}$. Thus, with $B_{\alpha\beta} = B(T_\alpha, T_\beta)$,

$$B_{\delta\epsilon} f^{(\epsilon)}_{(\alpha\beta)(\gamma)} + B_{\gamma\epsilon} f^{(\epsilon)}_{(\alpha\beta)(\delta)} = 0 . \quad (12)$$

From (9), this is the same as

$$B_{\delta\epsilon} \Gamma^\epsilon_\gamma + B_{\gamma\epsilon} \Gamma^\epsilon_\delta = 0 . \quad (13)$$

If B is a metric, so that we can lower indices, it becomes $\Gamma_{\delta\gamma} = -\Gamma_{\gamma\delta}$.

This is typical of (pseudo-)orthogonal connections, 1-forms with values in the Lie algebra of (pseudo-)orthogonal groups. Each bilinear form satisfying this condition gives an invariant metric. This is clearly the case for the de Sitter groups and for P a semisimple group in general: the Cartan–Killing metric will do. We know, on the other hand, that the Killing form is degenerate for non-semisimple groups. However, any other such bilinear can lead to a metric. It will be shown below how this happens for the Poincaré group case, for which the Lorentz metric will come up naturally.

When we say, rather loosely, that a metric exists, or not, we mean that there exists, or not, a metric which is invariant under the kinematical group of transformations. Thus, we shall see that there is no metric on Galilean “spacetime” which is invariant under the Galilei group, and no metric on the Newton–Hooke spacetime which is invariant under the Newton–Hooke transformations. The same invariance requirement holds for connections. In both cases above there will be invariant connections (though trivial in the Galilean case).

IV. THE DE SITTER CASES

We start by introducing the de Sitter spacetime $DS(4, 1)$ and the (anti-) de Sitter spacetime $DS(3, 2)$ as hypersurfaces in the pseudo–Euclidean spacetimes $\mathbf{E}^{4,1}$ and $\mathbf{E}^{3,2}$, inclusions whose points in Cartesian coordinates $(\xi) = (\xi^0, \xi^1, \xi^2, \xi^3, \xi^4)$ satisfy, respectively,

$$\begin{aligned} (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^0)^2 + (\xi^4)^2 &= L^2 ; \\ (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 - (\xi^0)^2 - (\xi^4)^2 &= -L^2 . \end{aligned}$$

We use $\eta_{\alpha\beta}$ for the Lorentz metric $\eta = \text{diag}(-1, 1, 1, 1)$ and the notation $\eta_{44} = \epsilon$ to put the conditions together as

$$\eta_{\alpha\beta} \xi^\alpha \xi^\beta + \epsilon (\xi^4)^2 = \epsilon L^2 .$$

We now change to stereographic coordinates $\{x^\mu\}$ in 4–dimensional space, which are given by

$$x^\mu = h_\alpha{}^\mu \xi^\alpha , \tag{14}$$

where

$$h^\alpha{}_\mu = n \delta^\alpha{}_\mu ; \tag{15}$$

$$n = \frac{1}{2} \left(1 - \frac{\xi^4}{L} \right) = \frac{1}{1 + \epsilon \sigma^2 / 4L^2} , \tag{16}$$

with $\sigma^2 = \eta_{\alpha\beta} \delta^\alpha{}_\mu \delta^\beta{}_\nu x^\mu x^\nu$. Calculating the line element for the de Sitter spacetimes, we find $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, where

$$g_{\mu\nu} = h^\alpha{}_\mu h^\beta{}_\nu \eta_{\alpha\beta} . \tag{17}$$

The $h^\alpha{}_\mu$ given in (15) are the components of the tetrad field, actually of the 1-form basis members

$$\omega^\alpha = h^\alpha{}_\mu dx^\mu = n \delta^\alpha{}_\mu dx^\mu , \tag{18}$$

dual to a vector basis

$$e_\alpha = h_\alpha{}^\mu \partial_\mu = \frac{1}{n} \delta_\alpha{}^\mu \partial_\mu . \tag{19}$$

The de Sitter spacetimes are conformally flat, with the conformal factor $n^2(x)$ given by (16). We could have written simply $x^\mu = \xi^\mu/n$, but we are carefully using the first letters of the Greek alphabet for the algebra (and flat space) indices, and those from the middle on for the homogeneous space fields and cofields. A certain fastidiousness concerning indices is necessary to avoid any confusion in the later contraction to the Newton–Hooke cases, and accounts for some apparently irrelevant δ 's in some expressions. As usual with changes from flat tangent–space to spacetime, letters of the two kinds are interchanged with the help of the tetrad field: for example, $f^{(\sigma)}_{(\alpha\beta)(\mu)} = h_\gamma{}^\sigma f^{(\gamma)}_{(\alpha\beta)(\epsilon)} h^\epsilon{}_\mu$. This is true for all tensor indices. Connections, which are vectors only in the last (1-form) index, acquire an extra “vacuum” term:

$$\Gamma^\lambda{}_{\mu\nu} = h_\alpha{}^\lambda \Gamma^\alpha{}_{\beta\nu} h^\beta{}_\mu + h_\gamma{}^\lambda \partial_\nu h^\gamma{}_\mu . \quad (20)$$

The Christoffel symbols corresponding to the metric $g_{\mu\nu}$ are

$$\Gamma^\lambda{}_{\mu\nu} = \left[\delta^\lambda{}_\mu \delta^\sigma{}_\nu + \delta^\lambda{}_\nu \delta^\sigma{}_\mu - g_{\mu\nu} g^{\lambda\sigma} \right] \partial_\sigma (\ln n) . \quad (21)$$

The Riemann tensor components, $R^\alpha{}_{\beta\rho\sigma} = \partial_\rho \Gamma^\alpha{}_{\beta\sigma} - \partial_\sigma \Gamma^\alpha{}_{\beta\rho} + \Gamma^\alpha{}_{\epsilon\rho} \Gamma^\epsilon{}_{\beta\sigma} - \Gamma^\alpha{}_{\epsilon\sigma} \Gamma^\epsilon{}_{\beta\rho}$, are found to be

$$R^\alpha{}_{\beta\rho\sigma} = \epsilon \frac{1}{L^2} \delta_\beta{}^\mu [\delta^\alpha{}_\rho g_{\mu\sigma} - \delta^\alpha{}_\sigma g_{\mu\rho}] . \quad (22)$$

The Ricci tensor and the scalar curvature are, consequently

$$R_{\mu\nu} = \epsilon \frac{3}{L^2} g_{\mu\nu} ; \quad (23)$$

$$R = \epsilon \frac{12}{L^2} . \quad (24)$$

Minkowski quantities are found in the contraction limit $L \rightarrow \infty$. Galilean kinematics comes then from the subsequent contraction $c \rightarrow \infty$. As usual with contractions, some infinities are absorbed in new, redefined, parameters. The Newton–Hooke cases are attained by taking the non–relativistic $c \rightarrow \infty$ limit while keeping finite an appropriate time parameter $\tau = L/c$ (see section V).

Let us now examine the Lie algebra of the de Sitter groups. The Lorentz sector is given by

$$[L_{\alpha\beta}, L_{\gamma\delta}] = \frac{1}{2} f^{(\epsilon\phi)}_{(\alpha\beta)(\gamma\delta)} L_{\epsilon\phi} , \quad (25)$$

with

$$f^{(\epsilon\phi)}_{(\alpha\beta)(\gamma\delta)} = \eta_{\beta\gamma} \delta^\epsilon{}_\alpha \delta^\phi{}_\delta + \eta_{\alpha\delta} \delta^\epsilon{}_\beta \delta^\phi{}_\gamma - \eta_{\beta\delta} \delta^\epsilon{}_\alpha \delta^\phi{}_\gamma - \eta_{\alpha\gamma} \delta^\epsilon{}_\beta \delta^\phi{}_\delta . \quad (26)$$

The de Sitter translation generators are Lorentz vectors,

$$[L_{\alpha\beta}, T_\gamma] = f^{(\epsilon)}_{(\alpha\beta)(\gamma)} T_\epsilon , \quad (27)$$

with

$$f^{(\epsilon)}_{(\alpha\beta)(\gamma)} = \eta_{\gamma\beta} \delta^\epsilon_\alpha - \eta_{\gamma\alpha} \delta^\epsilon_\beta . \quad (28)$$

Up to this point, of course, the situation is identical to that of the Poincaré group. For the invariant metric also, as only the values of $f^{(\epsilon)}_{(\alpha\beta)(\gamma)}$ are involved, the same bilinear will work for both de Sitter and Poincaré. For the translation sector, we shall have

$$[T_\alpha, T_\beta] = \frac{1}{2} f^{(\epsilon\phi)}_{(\alpha)(\beta)} L_{\epsilon\phi} + f^{(\gamma)}_{(\alpha)(\beta)} T_\gamma , \quad (29)$$

with

$$f^{(\epsilon\phi)}_{(\alpha)(\beta)} = -\frac{\epsilon}{L^2} \left(\delta^\epsilon_\alpha \delta^\phi_\beta - \delta^\phi_\alpha \delta^\epsilon_\beta \right) ; \quad (30)$$

$$f^{(\gamma)}_{(\alpha)(\beta)} = 0 . \quad (31)$$

The contraction factor $1/L^2$ has already been introduced in (30). In the contraction limit $L \rightarrow \infty$, it gives the usual commutative translations of the Poincaré group. The vanishing in (31) accounts for the absence of torsion in all cases we shall be concerned with.

Concerning the existence of an invariant metric, the condition (12) will be

$$\eta_{\gamma\beta} B_{\delta\alpha} - \eta_{\gamma\alpha} B_{\delta\beta} + \eta_{\delta\beta} B_{\gamma\alpha} - \eta_{\delta\alpha} B_{\gamma\beta} = 0 . \quad (32)$$

This is satisfied, in particular, by $B_{\alpha\beta} = \eta_{\alpha\beta}$, which is quite natural: the group is usually introduced by this condition. The usual condition defining an orthogonal group says that $BAB^{-1} = -A^T$, for A any member of the group Lie algebra. Take the generators $J_{\gamma\delta}$: the condition becomes $B_{\alpha\beta} (J_{\gamma\delta})^\beta_\epsilon B^{\epsilon\phi} = -(J_{\gamma\delta})^\phi_\alpha$ which implies $B_{\alpha\beta} f^{(\beta)}_{(\gamma\delta)(\epsilon)} B^{\epsilon\phi} = -f^{(\phi)}_{(\gamma\delta)(\alpha)}$. Lowering and raising indices with B and B^{-1} , we find $f_{(\alpha)(\gamma\delta)}^{(\phi)} = -f^{(\phi)}_{(\gamma\delta)(\alpha)}$, equivalent to (12). Thus, for (pseudo-)orthogonal groups, B can always be the original preserved metric. In the present case, (12) is also satisfied by any metric of the form $B_{\alpha\beta} = n^2 \eta_{\alpha\beta}$, with n^2 positive.

Using the fact that the metric $\eta_{\alpha\beta}$ is invariant under the action of the de Sitter group, we can use (13) and the transformation rule (20) to write

$$D_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\sigma_{\mu\lambda} g_{\sigma\nu} - \Gamma^\sigma_{\nu\lambda} g_{\mu\sigma} = 0 ,$$

which says that the de Sitter canonical connection $\Gamma^\lambda_{\mu\nu}$, with torsion

$$\Theta = -\frac{1}{2} T_\alpha f^{(\alpha)}_{(\gamma)(\delta)} \omega^\gamma \wedge \omega^\delta = 0 ,$$

parallel-transport the de Sitter spacetime metric $g_{\mu\nu}$ given in (17). According to Ricci's theorem¹⁴ there is a unique linear connection Γ which preserves a metric g and has a fixed Θ for its torsion. Therefore, we conclude that the canonical connection of the de Sitter spacetimes regarded as reductive homogeneous spacetimes is exactly the same as the de Sitter Christoffel symbols (21) obtained by requiring *a priori* the connection to be torsionless

and the covariant derivative of the metric (17) to be zero. Starting from (21), and using the transformation rule (20), we find

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{n} [\eta_{\alpha\gamma}\delta^\epsilon_\beta - \eta_{\beta\gamma}\delta^\epsilon_\alpha] e_\epsilon(n) , \quad (33)$$

with $e_\epsilon(n)$ the vector basis (19) applied to n .

This canonical connection has algebraic curvature with components $R^{\alpha\beta}_{\gamma\delta} = -f^{(\alpha\beta)}_{(\gamma)(\delta)}$, from which we obtain

$$R^\alpha_{\beta\gamma\delta} = \eta_{\beta\phi} R^{\alpha\phi}_{\gamma\delta} = \frac{\epsilon}{L^2} [\eta_{\beta\delta}\delta^\alpha_\gamma - \eta_{\beta\gamma}\delta^\alpha_\delta] . \quad (34)$$

The algebraic Ricci tensor will be

$$R_{\alpha\beta} = \epsilon \frac{3}{L^2} \eta_{\alpha\beta} . \quad (35)$$

By using the tetrad field, the components of the geometrical curvature can be obtained by

$$R^\alpha_{\beta\mu\nu} = R^\alpha_{\beta\gamma\delta} h_\gamma^\mu h_\delta^\nu .$$

We thus obtain (22), and subsequently (23) and (24).

It has been discussed in the literature whether the connection determines the metric and whether curvature determines the connection.¹⁵ In the case of the de Sitter spacetimes we can use the homotopy formula¹⁴ to obtain the connection from their Riemann tensor, and the metric from their connection. As this discussion is out of the scope of the present paper, we shall not present it here.

V. THE NEWTON–HOOKE SPACETIMES

Newton–Hooke spacetimes can be considered as the non–relativistic limits of the de Sitter ones. Their main characteristic is that the spacetime translations contain a global effect inherited from the de Sitter curvature. This is due to the fact that, in contrast to the Galilei group, no non–cosmological limit is taken to get the Newton–Hooke groups.

To obtain the Newton–Hooke algebras, we submit de Sitter multiplication table to a Inönü–Wigner contraction. In the contraction of a Lie algebra, we must ensure the desired behavior of the limiting generators through a careful choice of parameterization, which reflects itself in the structure coefficients. This is better understood if we take a particular representation. Let us consider the so–called kinematical representation, in which the de Sitter generators are given by vector fields on $\mathbf{E}^{3,2}$ and $\mathbf{E}^{4,1}$:

$$L_{AB} = \xi_A \frac{\partial}{\partial \xi^B} - \xi_B \frac{\partial}{\partial \xi^A} , \quad (36)$$

with the indices $A, B = 0, 1, 2, 3, 4$. We then pass to the stereographic coordinates (14), with the identifications

$$(x^i) = (x, y, z) : \text{Cartesian coordinates in } \mathbf{E}^3 ; \\ x^0 = ct .$$

From now on, we shall be using $a, b, c, \dots = 1, 2, 3$ for the algebra indices, and $i, j, k, \dots = 1, 2, 3$ for the space indices. The non-relativistic case is achieved in the limit $c \rightarrow \infty$, after appropriate redefinitions of quantities which would otherwise exhibit divergences. Here, we must first separate time and space components of L_{AB} , obtaining explicit forms for L_{ab} , L_{a0} , L_{a4} and L_{04} , and then redefine these operators so as to have finite expressions in the limit. In the present case, we must go through the intermediate step of introducing a new time parameter $\tau = L/c$ which remains finite in the process. The redefinitions are the following:

$$\mathbb{L}_{ab} \equiv L_{ab} \ , \ \mathbb{L}_{a0} \equiv L_{a0}/c \ , \ \mathbb{T}_a \equiv \epsilon L_{a4}/c\tau \ , \ \mathbb{T}_0 \equiv \epsilon L_{04}/\tau \ . \quad (37)$$

This corresponds to introducing the inverse factors in the parameters, so that $\omega^{ab} \rightarrow \omega^{ab}$; $\omega^{a0} \rightarrow c\omega^{a0}$; $\omega^a \rightarrow \epsilon c\tau\omega^a$, and $\omega^0 \rightarrow \epsilon\tau\omega^0$. The factors are then absorbed in the redefined parameters, which acquire different dimensions. Connections behave like parameters (more precisely: like the dual Maurer–Cartan forms, which behave like parameters), so that a connection component Γ^{a0} , for example, will acquire a factor $(1/c)$. The de Sitter conformal function (16) becomes

$$n(t) = \frac{1}{1 - \epsilon t^2/4\tau^2} \ , \quad (38)$$

and the tetrad fields will consequently be

$$h^\alpha{}_\mu = \frac{\delta^\alpha{}_\mu}{1 - \epsilon t^2/4\tau^2} \ . \quad (39)$$

In terms of the redefined generators, the de Sitter multiplication table (25)-(31) becomes

$$[\mathbb{L}_{ab}, \mathbb{L}_{de}] = \delta_{bd}\mathbb{L}_{ae} + \delta_{ae}\mathbb{L}_{bd} - \delta_{be}\mathbb{L}_{ad} - \delta_{ad}\mathbb{L}_{be} \ ; \quad (40)$$

$$[\mathbb{L}_{ab}, \mathbb{L}_{d0}] = \delta_{bd}\mathbb{L}_{a0} - \delta_{ad}\mathbb{L}_{b0} \ ; \quad (41)$$

$$[\mathbb{L}_{0b}, \mathbb{L}_{0e}] = \frac{1}{c^2}\mathbb{L}_{be} \ ; \quad (42)$$

$$[\mathbb{L}_{ab}, \mathbb{T}_d] = \delta_{bd}\mathbb{T}_a - \delta_{ad}\mathbb{T}_b \ ; \quad (43)$$

$$[\mathbb{L}_{a0}, \mathbb{T}_b] = \frac{1}{c^2}\delta_{ab}\mathbb{T}_0 \ ; \quad (44)$$

$$[\mathbb{L}_{a0}, \mathbb{T}_0] = -\mathbb{T}_a \ ; \quad (45)$$

$$[\mathbb{L}_{ab}, \mathbb{T}_0] = 0 \ ; \quad (46)$$

$$[\mathbb{T}_a, \mathbb{T}_b] = -\frac{\epsilon}{\tau^2 c^2}\mathbb{L}_{ab} \ ; \quad (47)$$

$$[\mathbb{T}_a, \mathbb{T}_0] = -\frac{\epsilon}{\tau^2} \mathbb{L}_{a0} ; \quad (48)$$

$$[\mathbb{T}_0, \mathbb{T}_0] = 0 . \quad (49)$$

This is the appropriate parameterization, in which the Newton–Hooke algebra is obtained by taking the limit $c \rightarrow \infty$. The result is then

$$[\mathbb{L}_{ab}, \mathbb{L}_{de}] = \delta_{bd} \mathbb{L}_{ae} + \delta_{ae} \mathbb{L}_{bd} - \delta_{be} \mathbb{L}_{ad} - \delta_{ad} \mathbb{L}_{be} ; \quad (50)$$

$$[\mathbb{L}_{ab}, \mathbb{L}_{d0}] = \delta_{bd} \mathbb{L}_{a0} - \delta_{ad} \mathbb{L}_{b0} ; \quad (51)$$

$$[\mathbb{L}_{0b}, \mathbb{L}_{0e}] = 0 ; \quad (52)$$

$$[\mathbb{L}_{ab}, \mathbb{T}_d] = \delta_{bd} \mathbb{T}_a - \delta_{ad} \mathbb{T}_b ; \quad (53)$$

$$[\mathbb{L}_{a0}, \mathbb{T}_b] = 0 ; \quad (54)$$

$$[\mathbb{L}_{a0}, \mathbb{T}_0] = -\mathbb{T}_a ; \quad (55)$$

$$[\mathbb{L}_{ab}, \mathbb{T}_0] = 0 ; \quad (56)$$

$$[\mathbb{T}_a, \mathbb{T}_b] = 0 ; \quad (57)$$

$$[\mathbb{T}_a, \mathbb{T}_0] = -\frac{\epsilon}{\tau^2} \mathbb{L}_{a0} ; \quad (58)$$

$$[\mathbb{T}_0, \mathbb{T}_0] = 0 . \quad (59)$$

Vanishing torsion is inherited from the de Sitter cases. We see from (58) an important difference with respect to the Galilean case: time translations do not commute with space translations. In consequence, there will be constants surviving contraction in the curvature (10). Such non-vanishing components of the algebraic curvature $R^{\alpha\beta}{}_{\gamma\delta}$ will be of the form

$$R^{a0}{}_{b0} = \frac{\epsilon}{\tau^2} \delta^a{}_b , \quad (60)$$

and those obtained by antisymmetrizing in the index-pairs. We obtain the non-vanishing geometrical curvature components $R^{\alpha\beta}{}_{\mu\nu}$ from (60) by using the tetrad field (39):

$$R^{a0}{}_{i0} = -R^{a0}{}_{0i} = -R^{0a}{}_{i0} = R^{0a}{}_{0i} = \frac{\epsilon}{\tau^2} \delta^a{}_i n^2(t) .$$

Thus, the Newton–Hooke “spacetimes” are curved, in the sense that their canonical connections as homogeneous spaces have non–vanishing curvature Riemann tensor.

To show the absence of metric we notice that, of all the constants appearing in condition (12), the only ones surviving contraction are $f^{(a)}_{(de)(b)} = (\delta_{eb}\delta_d^a - \delta_{db}\delta_e^a)$ and $f^{(a)}_{(b0)(0)} = -\delta^a_b$. Then, from

$$B_{\delta a}f^{(a)}_{(\alpha\beta)(\gamma)} + B_{\gamma a}f^{(a)}_{(\alpha\beta)(\delta)} = 0 ,$$

a component–by–component analysis shows that B must have $B_{ab} = 0$ and $B_{0b} = 0$. There is no condition on the component B_{00} , which can assume any value. Anyhow, the bilinear form will be degenerate. No metric is possible. There is no spacetime in the usual, metric sense.

The procedure extends immediately to the Galilei case. Galilean spacetime is, however, flat in the same sense in which the Newton–Hooke spacetime is curved: the Galilean canonical connection has vanishing Riemann tensor. Notice that to have a flat connection is quite different from having no connection at all. Galilean connection is flat, but it exists.

For completeness we have calculated explicitly the canonical connection of Newton–Hooke spacetimes. Taking the appropriate limit of the de Sitter canonical connection (33), we obtain the non–vanishing components of the Newton–Hooke connection $\Gamma^{\alpha\beta}(x)$, namely

$$\Gamma^{a0}(x) = -\Gamma^{0a}(x) = -\frac{\epsilon}{2\tau^2} \delta^a_i \left[t dx^i - x^i dt \right] n(t) . \quad (61)$$

The Newton–Hooke spacetimes appear then as examples of non–metric curved spacetimes. Despite the non–relativistic limit, the effects of curvature are present due to the fact that we are still considering a kinematical spacetime on a large scale of time. Notwithstanding, in such a spacetime there is an absolute time. As in the Galilean case, simultaneity of two events is preserved by a general inertial transformation. Another interesting physical feature of Newton–Hooke spacetimes is that, analogously to the de Sitter cases, energy is not invariant under spatial translations. This can be directly verified from Eqs. (48) and (58).

The explicit form of the infinitesimal generators for rotations and boosts are the same for Newton–Hooke and Galilei groups. For the spacetime translations generators, Newton–Hooke differ from Galilei by factors proportional to τ^{-2} , which vanish in the $\tau \rightarrow \infty$ limit.

VI. FINAL REMARKS

Newton–Hooke groups and spacetimes appear as the non–relativistic limits in all theories involving de Sitter groups and spacetimes, in the same way the Galilei group appears as the corresponding non–cosmological non–relativistic limit. Therefore, if the relativistic spacetime kinematical group were supposed to be de Sitter instead of Poincaré, the non–relativistic limit would lead to Newton–Hooke instead of Galilei, the latter being obtained through a further non–cosmological contraction. It is commonly accepted that non–relativistic physics is invariant under the Galilei group, and it is known since long that the cosmological constant is very small inside the solar system. Discrepancies between values of the Hubble constant obtained from nearby and distant objects have led, however, to the proposal that the “local”

values (cosmologists say “small scale” values) of cosmological quantities are not typical, and could be different in other regions of the Universe.¹⁶ Newton–Hooke spacetimes are, thus, fair candidates for non-relativistic physical spacetimes in regions in which the value of τ is large.

The main differences of Newton–Hooke with respect to Galilei kinematics come from the non-commutativity between space and time translations. Usual non-relativistic time-evolution equations, for example, in which the 3-space Laplacian is time-invariant, would be changed. Local Newton–Hooke spacetimes can, therefore, lead to important modifications in cosmological non-relativistic physics, as used in the approach to galaxy clusters via the virial theorem, or in the study of the rotation spectra of individual galaxies.

There are further, more conceptual aspects. In its standard definition, a spacetime is a 4-manifold with a Lorentzian metric g . More precisely, it is a pair (\mathcal{M}, g) where \mathcal{M} is a connected 4-dimensional differentiable manifold and g is a metric of signature 2 on \mathcal{M} . The metric provides a natural torsionless Levi–Civita connection, with a Riemann curvature associated to it, but the fact remains that curvature is a connection, metric-independent, characteristic. The existence of curved spacetimes on which no metric is defined suggests that “spacetime” should be defined not as a pair (\mathcal{M}, g) , a manifold plus a metric, but as a pair (\mathcal{M}, Γ) — a differentiable manifold plus a connection. Galilean spacetime would in that case acquire a well-defined status, despite its flat connection. Gravitation would be brought closer to the other fundamental interactions of Nature, all of them described by connections. In the standard cases of metric spacetimes, the connection would be the Christoffel symbols obtained by requiring it to parallel-transport the metric and to have zero torsion. A connection satisfying these two conditions is unique, according to Ricci’s theorem. Consequently, when spacetime is also a homogeneous space, this connection coincides with the torsionless canonical connection obtained following the general algebraic scheme of section 2, which complies with both of the above conditions.

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